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Relativistic formulation of quantum-state diffusion

Heinz-Peter Breuer and Francesco Petruccione

Albert-Ludwigs-Universität, Fakultät für Physik, Hermann-Herder Straße 3, D-79104 Freiburg im Breisgau, Germany

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Abstract. A relativistic generalization of the quantum-state diffusion model is developed. The model describes a Dirac electron which is coupled to an external electromagnetic field and a dissipative environment. A relativistically covariant stochastic Dirac equation is obtained by regarding the state vector as a functional on a certain set of spacelike hypersurfaces in Minkowski space and by the definition of an appropriate Hilbert bundle on this set. The integrability condition of the stochastic process and the corresponding covariant density matrix equation are derived. Further, the relativistic equations governing the dynamical state-vector localization are deduced.

1. Introduction

The model of quantum-state diffusion [1–4] represents the time-evolution of an open quantum system as a stochastic diffusion process in Hilbert space. It has been designed in order to describe the dynamics of individual quantum systems under the influence of a dissipative environment. In particular, it describes the localization or state-vector reduction during a measurement as a dynamical process [5–7].

To recall briefly the basic equation of quantum-state diffusion, consider an open quantum system the density matrix of which is governed by an equation of motion in Lindblad form [8],

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\mathrm{i}[H, \rho] + L\rho L^{\dagger} - \tfrac{1}{2}L^{\dagger}L\rho - \tfrac{1}{2}\rho L^{\dagger}L \tag{1}$$

where we have chosen units such that $\hbar = 1$, H is the Hamiltonian and L some Lindblad operator which describes the dissipation mechanism. Since the generalization to several Lindblad operators is obvious, we restrict ourselves in this paper to the case of one Lindblad operator L. In quantum-state diffusion one replaces the density-matrix equation (1) by the following stochastic differential equation for the random state vector φ,

$$\mathrm{d}_t\varphi = -\mathrm{i}H\varphi\mathrm{d}t + (\langle L^{\dagger}\rangle L - \tfrac{1}{2}L^{\dagger}L - \tfrac{1}{2}\langle L^{\dagger}\rangle\langle L\rangle)\varphi\,\mathrm{d}t + (L - \langle L\rangle)\varphi\,\mathrm{d}W\,(t) \tag{2}$$

where $\mathrm{d}_t\varphi = \varphi(t + \mathrm{d}t) - \varphi(t)$, $\langle L\rangle = \langle\varphi|L|\varphi\rangle$ denotes the quantum expectation value, and $\mathrm{d}W(t)$ is the differential of a complex Wiener process. From equation (2) the density-matrix equation (1) is recovered by defining the density-matrix as the covariance matrix of the process,

$$\rho = E\{|\varphi\rangle\langle\varphi|\} \tag{3}$$

where E denotes the expectation value of the process.

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As it stands, equation (2) is a non-relativistic state-vector equation. The central aim of this paper is to demonstrate that the stochastic formulation of open quantum systems within the quantum-state diffusion model can be extended to relativistic quantum mechanics. In other words, we shall show that the equations of quantum-state diffusion can be cast into a form which is covariant under Lorentz transformations. From a fundamental point of view this is an important issue since Lorentz covariance implies that the stochastic formulation is in agreement with the relativity principle. In order to be specific we consider in the following a single Dirac electron in an external electromagnetic field as the quantum system. Thereby we assume that the external fields are so weak that a sensible one-particle description is justified. Our aim is thus to construct a stochastic Dirac equation which takes the form of a covariant generalization of equation (2).

The basic idea underlying our construction of a relativistic quantum-state-diffusion theory is quite similar to that of the Schwinger–Tomonaga formulation of relativistic quantum-field theory [9]. In an attempt to write the Schrödinger equation for the state vector Ψ of a quantum field in a covariant form, Schwinger and Tomonaga regarded Ψ as a functional $\Psi = \Psi[\sigma]$ on the set of spacelike hypersurfaces σ in Minkowski space. This ansatz leads to the following manifestly covariant state-vector equation, the so-called Schwinger–Tomonaga equation,

$$\frac{\delta\Psi[\sigma]}{\delta\sigma(x)} = -\mathrm{i}\boldsymbol{H}(x)\Psi[\sigma] \tag{4}$$

where $\boldsymbol{H}(x)$ denotes the Hamiltonian density of the theory at the spacetime point x. We shall follow a similar strategy in this paper: we consider below the Dirac wavefunction ψ for a single electron not as a function on the spacetime continuum but rather as a functional on a certain set of spacelike hypersurfaces. A similar idea has been proposed in [10] to develop a covariant formulation of non-local quantum measurements.

The Schwinger–Tomonaga formulation deals, of course, with a closed quantum system. Our task will therefore be to couple, in a Lorentz covariant manner, additional dissipative and stochastic terms to an equation of the above type. It will be shown that this can, in fact, be done in such a way that a relativistic quantum-state-diffusion equation is obtained. Our formulation will be very general since it allows the use of Lindblad operators of any form. In particular it enables us to include local as well as non-local operators.

The paper is organized as follows. In section 2 we introduce an appropriate parametrization for the set of flat, spacelike hypersurfaces that will be used in our construction. Each hypersurface is equipped with a covariant scalar product making the space of Dirac wavefunctions on each hypersurface a Hilbert space. Further, we formulate the time evolution according to Dirac's equation in a manner similar to the Schwinger–Tomonaga equation (4).

Section 3 deals with our construction of a covariant quantum-state-diffusion equation for the Dirac electron. Its most important mathematical and physical properties are discussed. In particular, we derive the corresponding covariant density-matrix equations and demonstrate the normalization of the state vector. Our parametrization of the hypersurfaces in Minkowski space singles out a specific point which, for simplicity, is chosen to be the origin of the coordinate system. Restricting ourselves first to homogeneous Lorentz transformations this point is left invariant and a theory which is covariant under homogeneous Lorentz transformations is obtained. In order to cope with spacetime translations this distinguished point will be given an objective physical meaning as an event in spacetime connected with the preparation process of the initial condition. It will be shown that a stochastic Dirac equation is then obtained which is covariant under inhomogeneous Lorentz transformations.

In our formulation of covariance the stochastic time-evolution of the state vector is, therefore, closely connected with the initial preparation event and its invariant future light cone. Finally, we derive the covariant equations governing the dynamical localization of the state vector.

Section 4 contains a summary and our conclusions.

2. The Dirac equation on the set of flat, spacelike hypersurfaces

In this section we shall introduce our basic notions and, in particular, the phase space on which the stochastic process of relativistic quantum-state diffusion is defined. The space Σ of all flat, spacelike hypersurfaces σ in Minkowski space will be introduced in section 2.1. On using the Dirac current one can define on each spacelike hypersurface σ a Hermitian scalar product. With this scalar product the set of all Dirac wavefunctions on a given hypersurface σ becomes a Hilbert space. The set of all these Hilbert spaces forms a so-called Hilbert bundle (section 2.2). It will then be shown that the Dirac equation defines a unitary evolution equation in this Hilbert bundle (section 2.3). Moreover, the condition that follows from the integrability of the obtained evolution equation will be derived.

2.1. Flat, spacelike hypersurfaces in Minkowski space

In the following a point in Minkowski space $\mathbb{R}^4$ will be denoted by $x = (x^\mu) = (x^0, x^i) = (x^0, \boldsymbol{x})$, where greek indices run from 0 to 3, and latin indices from 1 to 3. The Lorentz scalar (inner) product is then given by $x \cdot y = x^\mu y_\mu = x^0 y^0 - \boldsymbol{x} \cdot \boldsymbol{y}$. Throughout this paper we choose units such that $c = \hbar = 1$. Lorentz transformations are written as

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu. \tag{5}$$

For the sake of clarity we shall consider here transformations Λ in the subgroup of homogeneous, proper, orthochronous Lorentz transformations. The generalization to inhomogeneous Lorentz transformations and the proof of Poincaré covariance will be given in section 3.5.

Consider the interior F_+ of the forward (future) light cone based at the origin of a given coordinate system. Clearly, the points of F_+ are given by the conditions

$$x^2 \equiv (x^0)^2 - \boldsymbol{x} \cdot \boldsymbol{x} > 0 \qquad x^0 > 0. \tag{6}$$

Each flat, spacelike hypersurface σ in Minkowski space which crosses F_+ can then be characterized uniquely by the following equation

$$n^\mu x_\mu = a. \tag{7}$$

Here, n is a unit 4-vector normal to the hypersurface σ,

$$n^2 = n^\mu n_\mu = 1 \qquad n^0 > 0 \tag{8}$$

and a is a positive Lorentz invariant scalar which represents the Lorentz distance of σ to the origin. Note that n is a timelike vector since the hypersurface is assumed to be spacelike. Thus, each hypersurface σ that crosses F_+ is uniquely defined by its unit normal 4-vector n and by a Lorentz invariant scalar a.

Alternatively, we can define the 4-vector $s = an$ and rewrite equation (7) as

$$s^\mu x_\mu = s^2. \tag{9}$$

We then have $s^2 = a^2 > 0$ and $s^0 > 0$. The 4-vector s therefore lies within the interior of the future light cone F_+. In the following we shall use both parametrizations of the flat,

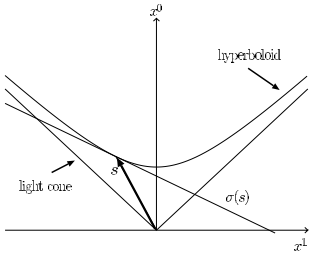

Figure 1. Geometrical construction of the hypersurface $\sigma(s)$.

spacelike hypersurfaces. When using the parametrization by n and a we write $\sigma = \sigma(n, a)$, when using the parametrization by the 4-vector s we write $\sigma = \sigma(s)$. Clearly, we have $\sigma(n, a) = \sigma(s)$ since $s = na$. We remark that the hypersurface $\sigma(s)$ is the tangent space of the hyperboloid defined by the equation $x^2 = s^2$ at the point $x = s$. Using this fact it is easy to visualize the geometrical construction of the hypersurface $\sigma(s)$ (see figure 1).

The set of all hypersurfaces $\sigma(s)$ will be denoted by Σ. The physical relevance of this space for our construction stems from the following facts. Since the forward light cone F_+ is mapped onto itself under Lorentz transformations Λ, also the set Σ is mapped onto itself under these transformations. That is, if σ belongs to Σ, then also $\Lambda\sigma$ belongs to Σ. In the special case $n = (1, 0, 0, 0)$ the corresponding hypersurface is simply the three-dimensional position space at fixed time $x^0 = t = a$. Since any n can be transformed into $(1, 0, 0, 0)$ by a suitable Lorentz transformation, the space Σ introduced above consists of those hypersurfaces that appear as ordinary 3-space at some fixed, positive time in some Lorentz frame. Thus, Σ is the space of all instantaneous hypersurfaces (with positive time coordinate) in all inertial frames. As will be shown in section 3, on the basis of this construction a natural relativistically covariant formulation of the quantum-state-diffusion model emerges.

As already mentioned, the above construction is similar to the well-known Schwinger–Tomonaga formulation [9] of relativistic quantum-field theory. For the reason explained above we restrict ourselves, however, to *flat* hypersurfaces. Being much simpler technically, this restriction turns out to be fully sufficient for our purpose.

Of course, by restricting ourselves to homogeneous Lorentz transformations we have in the above construction distinguished a specific point, namely the base point $x = 0$ of the future light cone. This restriction will be removed in section 3.5 where we shall give a formulation of quantum-state diffusion that is covariant under Poincaré transformations.

2.2. The Hilbert bundle

With each hypersurface $\sigma(s)$ we can associate a Hilbert space $\mathcal{H}(s)$ of Dirac wavefunctions in the following way. Each point $x \in \sigma(s)$ has a unique representation of the form $x = (x^0(s, \boldsymbol{x}), \boldsymbol{x})$, where $\boldsymbol{x} \in \mathbb{R}^3$ and the time coordinate is given by

$$x^0(s, \boldsymbol{x}) = \frac{\boldsymbol{s} \cdot \boldsymbol{x} + s^2}{s^0} = \frac{\boldsymbol{n} \cdot \boldsymbol{x} + a}{n^0}. \tag{10}$$

On each hypersurface $\sigma(s)$ we can then introduce a Dirac wavefunction ψ by writing

$$\psi(x) = \psi(s, \boldsymbol{x}). \tag{11}$$

This equation means that $\psi(s, \boldsymbol{x})$ is the value of the Dirac wavefunction ψ taken at the point $x = (x^0(s, \boldsymbol{x}), \boldsymbol{x}) \in \sigma(s)$.

In order to make the set of Dirac wavefunctions on a fixed hypersurface $\sigma(s)$ a Hilbert space we need to define a positive definite Hermitian scalar product in this space. To this end, we observe first that the Lorentz metric induces a positive, three-dimensional volume element $\mathrm{d}\sigma$ on each hypersurface given by

$$\mathrm{d}\sigma = \frac{1}{n^0}\,\mathrm{d}^3x. \tag{12}$$

Since n is the unit vector normal to $\sigma(s)$ the 4-vector $n_\mu \mathrm{d}\sigma$ is the surface element of the hypersurface. On using the Dirac current

$$j^\mu = \bar{\psi}\gamma^\mu\psi \tag{13}$$

where $\bar{\psi} = \psi^\dagger\gamma^0$ and the γ^μ denote the usual gamma matrices (we use the notation of [11]), we can form the Lorentz invariant bilinear expression

$$\langle\psi|\psi\rangle_s \equiv \int_\sigma j^\mu n_\mu \,\mathrm{d}\sigma. \tag{14}$$

Physically this expression represents the probability flow through the hypersurface $\sigma(s)$ and suggests the following definition of a Hermitian scalar product:

$$\langle\psi|\phi\rangle_s \equiv \int_\sigma \mathrm{d}\sigma\,\bar{\psi}\not{n}\phi = \int \frac{\mathrm{d}^3x}{n^0}\,\psi^\dagger(s,\boldsymbol{x})\gamma^0\not{n}\phi(s,\boldsymbol{x}) \tag{15}$$

where we define $\not{n} \equiv n_\mu\gamma^\mu$. It is easy to show that the expression (15) has the properties of a Hermitian scalar product. In particular it is a positive definite Hermitian form. This is a direct consequence of the fact that the (4×4)-matrix $\gamma^0\not{n}$ is positive definite. Note further that for the special case $n = (1,0,0,0)$, that is, for $s = (a,0,0,0)$ definition (15) reduces to the usual scalar product [11] taken at a fixed time $x^0 = t = a$,

$$\langle\psi|\phi\rangle_{(a,0,0,0)} = \int \mathrm{d}^3x\;\psi^\dagger(t,\boldsymbol{x})\phi(t,\boldsymbol{x}). \tag{16}$$

The space of Dirac spinors on $\sigma(s)$ with finite norm

$$||\psi|| \equiv \sqrt{\langle\psi|\psi\rangle_s} \tag{17}$$

forms a Hilbert space which will be denoted by $\mathcal{H}(s)$. Thus, we have associated with each hypersurface a corresponding Hilbert space of Dirac wavefunctions,

$$\sigma(s) \mapsto \mathcal{H}(s). \tag{18}$$

Note that, in mathematical terms, the mapping (18) defines a so-called Hilbert bundle. Its base manifold consists of the various hypersurfaces $\sigma(s)$ and its fibres are given by the Hilbert spaces $\mathcal{H}(s)$ attached to these surfaces.

2.3. Unitary time-evolution according to Dirac's equation

We now turn to the description of the dynamics according to the Dirac equation

$$(\mathrm{i}\gamma^\mu[\partial_\mu + \mathrm{i}eA_\mu(x)] - m)\Phi(x) = 0 \tag{19}$$

for an electron with charge e and mass m in an arbitrary external electromagnetic field described by the 4-vector potential $A_\mu(x)$. Any solution of equation (19) defines a Dirac wavefunction $\Phi = \Phi(x)$ on the spacetime continuum. Our aim is to reformulate the Dirac equation as a differential equation in the variables s^μ which parametrize the hypersurfaces $\sigma(s)$. The resulting equation then describes how the Dirac wavefunction changes when going from one hypersurface to another. To this end, we first write equation (19) as

$$\partial_0\Phi = -\mathrm{i}\gamma^0(-\mathrm{i}\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + e\not{A}(x) + m)\Phi \equiv -\mathrm{i}H_\mathrm{D}\Phi \tag{20}$$

where H_{D} denotes the Dirac Hamiltonian. Restricting the solution $\Phi(x)$ to some hypersurface $\sigma(s)$ we obtain the Dirac wavefunction on that surface. We therefore have

$$\psi(s,\boldsymbol{x}) = \Phi(x^0(s,\boldsymbol{x}),\boldsymbol{x}). \tag{21}$$

Differentiating this equation with respect to s^μ and using the Dirac equation (20) one finds

$$\frac{\partial}{\partial s^\mu}\psi(s,\boldsymbol{x}) = \frac{\partial x^0}{\partial s^\mu}(\partial_0\Phi)(x) = -\mathrm{i}\frac{\partial x^0}{\partial s^\mu}H_{\mathrm{D}}\Phi(x^0,\boldsymbol{x}). \tag{22}$$

Note that in the last equation the differential operator $\boldsymbol{\nabla}$ contained in H_{D} acts only on the second argument $\boldsymbol{x}$ of Φ. Since

$$(\partial_i\Phi)(x^0,\boldsymbol{x}) = \partial_i\Phi(x^0,\boldsymbol{x}) - \frac{\partial x^0}{\partial x^i}(\partial_0\Phi)(x^0,\boldsymbol{x}) = \partial_i\psi(s,\boldsymbol{x}) - \frac{s^i}{s^0}(\partial_0\Phi)(x^0,\boldsymbol{x}) \tag{23}$$

where $(\partial_i\Phi)$ and $(\partial_0\Phi)$ denote partial and $\partial_i\Phi$ total derivatives, we find

$$\begin{aligned} H_{\mathrm{D}}\Phi(x^0,\boldsymbol{x}) &= \gamma^0(-\mathrm{i}\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + e\not{A}(x) + m)\Phi(x^0,\boldsymbol{x}) \\ &= \gamma^0(-\mathrm{i}\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + e\not{A}(x) + m)\psi(s,\boldsymbol{x}) + \gamma^0\left(+\mathrm{i}\boldsymbol{\gamma}\cdot\frac{\boldsymbol{s}}{s^0}(\partial_0\Phi)(x^0,\boldsymbol{x})\right) \\ &= H_{\mathrm{D}}\psi(s,\boldsymbol{x}) + \mathrm{i}\gamma^0\boldsymbol{\gamma}\cdot\frac{\boldsymbol{s}}{s^0}(\partial_0\Phi)(x^0,\boldsymbol{x}) \\ &= \mathrm{i}(\partial_0\Phi)(x^0,\boldsymbol{x}). \end{aligned} \tag{24}$$

From the last equation we obtain

$$\left(I - \gamma^0\boldsymbol{\gamma}\cdot\frac{\boldsymbol{s}}{s^0}\right)(\partial_0\Phi) = -\mathrm{i}H_{\mathrm{D}}\psi(s,\boldsymbol{x}) \tag{25}$$

where I denotes the unit matrix. On multiplying this equation by $s^0\gamma^0$ we find

$$\not{s}(\partial_0\Phi) = -\mathrm{i}s^0\gamma^0 H_{\mathrm{D}}\psi(s,\boldsymbol{x}). \tag{26}$$

Now, multiply both sides by $\not{s}$, use the fact that $\not{s}\not{s} = s^2$, and divide by s^2. This yields the following equation for the time derivative of Φ,

$$(\partial_0\Phi)(x^0,\boldsymbol{x}) = -\mathrm{i}\frac{s^0\not{s}}{s^2}\gamma^0 H_{\mathrm{D}}\psi(s,\boldsymbol{x}). \tag{27}$$

Inserting this expression into equation (22) one obtains

$$\frac{\partial}{\partial s^\mu}\psi(s,\boldsymbol{x}) = -\mathrm{i}\left(\frac{s^0}{s^2}\frac{\partial x^0}{\partial s^\mu}\right)\not{s}\gamma^0 H_{\mathrm{D}}\psi(s,\boldsymbol{x}). \tag{28}$$

Using equation (10) it is easy to verify that

$$s^0\frac{\partial x^0}{\partial s^\mu} = 2s_\mu - x_\mu. \tag{29}$$

Thus, we are finally led to the following differential equation

$$\frac{\partial}{\partial s^\mu}\psi(s,\boldsymbol{x}) = -\mathrm{i}V_\mu K(s)\psi(s,\boldsymbol{x}) \tag{30}$$

where we have defined the operators in $\mathcal{H}(s)$

$$V_\mu \equiv V_\mu(s) \equiv \frac{1}{s^2}(2s_\mu - x_\mu) \tag{31}$$

$$K(s) \equiv \not{s}\gamma^0 H_{\mathrm{D}} = \not{s}(-\mathrm{i}\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + e\not{A}(x^0(s,\boldsymbol{x}),\boldsymbol{x}) + m). \tag{32}$$

Equation (30) which we call transport equation is just a reformulation of the Dirac equation (20) which represents an evolution equation in our Hilbert bundle. It describes

the change of the Dirac wavefunction when going from one flat, spacelike hypersurface to another. As will be demonstrated in the next section equation (30) is an appropriate starting point for the formulation of a relativistic theory of quantum-state diffusion.

To understand equation (30) in more detail it is helpful to write it in terms of our (n, a)-parametrization of the hypersurfaces given in equation (7). This is done most easily by employing the relation

$$\frac{\partial}{\partial s^\mu} = \frac{1}{a}(\delta^\nu_\mu - n_\mu n^\nu)\frac{\partial}{\partial n^\nu} + n_\mu \frac{\partial}{\partial a}. \tag{33}$$

This relation enables one to split equation (30) into two separate equations

$$\frac{\partial}{\partial a}\psi(n, a, \boldsymbol{x}) = -\mathrm{i}H(s)\psi(n, a, \boldsymbol{x}) \tag{34}$$

$$(\delta^\nu_\mu - n_\mu n^\nu)\frac{\partial}{\partial n^\nu}\psi(n, a, \boldsymbol{x}) = -\mathrm{i}W_\mu H(s)\psi(n, a, \boldsymbol{x}) \tag{35}$$

where we have defined the operators

$$H(s) \equiv \frac{1}{a}K(s) = \not{n}\gamma^0 H_\mathrm{D} \tag{36}$$

$$W_\mu \equiv W_\mu(s) = s_\mu - x_\mu. \tag{37}$$

Equation (34) describes how the Dirac wavefunction changes under parallel translations of the hypersurfaces. Note that if one considers the family of equal-time surfaces parametrized by $s = (a, 0, 0, 0)$, one has $H(s) = H_\mathrm{D}$ and, therefore, equation (34) reduces to the Dirac equation (20). Equation (35) represents the change of the wavefunction due to variations of the normal vector n of the hypersurfaces. These variations may be realized by varying the tip of the vector s along the hyperboloid depicted in figure 1. Note that the tensor $\delta^\nu_\mu - n_\mu n^\nu$ on the left-hand side of equation (35) acts as an orthogonal projection onto the hypersurface $\sigma(n, a)$. Consequently, the 4-vector W_μ (which has to be regarded as an operator in $\mathcal{H}(s)$) must satisfy the condition $n^\mu W_\mu = 0$, which is easily verfied.

It can be shown that the operators introduced in equations (31), (32), (36) and (37) are Hermitian with respect to the scalar product (15). This fact is obviously true for the operators V_μ and W_μ since they are diagonal in spin space. To prove the hermiticity of $K(s)$ one uses the fact that the Dirac operator is Hermitian in $\mathcal{H}(a, 0, 0, 0)$,

$$\int \mathrm{d}^3x\, \psi^\dagger H_\mathrm{D}\phi = \int \mathrm{d}^3x\, (H_\mathrm{D}\psi)^\dagger \phi. \tag{38}$$

Therefore, we find on using (32) and $\not{s}^\dagger\gamma^0 = \gamma^0\not{s}$

$$\begin{aligned}\langle\psi|K\phi\rangle_s &= \int \frac{\mathrm{d}^3x}{s^0}\bar{\psi}\not{s}K\phi = s^2\int \frac{\mathrm{d}^3x}{s^0}\psi^\dagger H_\mathrm{D}\phi \\ &= s^2\int \frac{\mathrm{d}^3x}{s^0}(H_\mathrm{D}\psi)^\dagger\phi = \int \frac{\mathrm{d}^3x}{s^0}(K\psi)^\dagger\gamma^0\not{s}\phi \\ &= \langle K\psi|\phi\rangle_s.\end{aligned}$$

This proves the hermiticity of $K(s)$ and also that of $H(s)$ since $H(s) = K(s)/a$.

The operator $V_\mu K$ appearing on the right-hand side of our transport equation (30) is not Hermitian, since V_μ and $K(s)$ do not commute. At first glance it might therefore seem that the transport equation does not preserve the norm of the wavefunction. However, one has to take care of the fact that by varying s also the scalar product changes. We now demonstrate that, taking into account the s-dependence of the scalar product, the transport equation is,

in fact, norm-conserving and thus defines a unitary evolution in the Hilbert bundle. Using the definition of the scalar product we obtain

$$\frac{\partial}{\partial s^\mu}\langle\psi|\psi\rangle_s = \left\langle\psi\left|\frac{\partial}{\partial s^\mu}\psi\right.\right\rangle_s + \left\langle\frac{\partial}{\partial s^\mu}\psi\right|\left.\psi\right\rangle_s + \int\frac{\mathrm{d}^3x}{s^0}\psi^\dagger\gamma^0\gamma_\mu\psi - \frac{1}{s^0}\frac{\partial s^0}{\partial s^\mu}\langle\psi|\psi\rangle_s. \tag{39}$$

The last two terms on the right-hand side stem from the s-dependence of the scalar product. Since V_μ and $K(s)$ are Hermitian we can write

$$\left\langle\psi\left|\frac{\partial}{\partial s^\mu}\psi\right.\right\rangle_s + \left\langle\frac{\partial}{\partial s^\mu}\psi\right|\left.\psi\right\rangle_s = -\mathrm{i}\langle\psi|[V_\mu, K]|\psi\rangle_s. \tag{40}$$

As can be easily verified the following commutation relation holds

$$[V_\mu, K] = -\frac{\mathrm{i}}{s^2}\not{s}\gamma_\mu + \frac{\mathrm{i}}{s^0}\delta_{0,\mu}. \tag{41}$$

Thus we get

$$\begin{aligned}\left\langle\psi\left|\frac{\partial}{\partial s^\mu}\psi\right.\right\rangle_s + \left\langle\frac{\partial}{\partial s^\mu}\psi\right|\left.\psi\right\rangle_s &= -\mathrm{i}\left\langle\psi|-\frac{\mathrm{i}}{s^2}\not{s}\gamma_\mu + \frac{\mathrm{i}}{s^0}\delta_{0,\mu}|\psi\right\rangle_s \\ &= -\int\frac{\mathrm{d}^3x}{s^0}\psi^\dagger\gamma^0\gamma_\mu\psi + \frac{1}{s^0}\frac{\partial s^0}{\partial s^\mu}\langle\psi|\psi\rangle_s.\end{aligned} \tag{42}$$

Looking at equation (39) we see that the additional terms that arise due to the s-dependent scalar product are cancelled by the terms on the right-hand side of equation (42) which stem from the non-vanishing commutator (41). Thus we have

$$\frac{\partial}{\partial s^\mu}\langle\psi|\psi\rangle_s = 0. \tag{43}$$

The transport equation (30) is a partial differential equation in four variables s^μ. The usual conditions for the integrabilty of such an equation can be written as

$$\frac{\partial^2\psi}{\partial s^\nu\partial s^\mu} = \frac{\partial^2\psi}{\partial s^\mu\partial s^\nu}. \tag{44}$$

We know, however, by construction that our transport equation is integrable. Given some initial condition on a specific hypersurface, its solution is given by (21), where Φ is the corresponding solution of the Dirac equation. Thus, inserting the transport equation into (44) we get the following important relation which may also be verified by an explicit calculation:

$$\frac{\partial}{\partial s^\nu}(V_\mu K) - \frac{\partial}{\partial s^\mu}(V_\nu K) - \mathrm{i}[V_\mu K, V_\nu K] = 0. \tag{45}$$

This relation will be used later on when we will deal with the integrabilty condition of the stochastic process of relativistic quantum-state diffusion.

3. Relativistic quantum-state diffusion

We demonstrate in this section that the construction of the preceding section enables us to formulate a relativistically covariant quantum-state diffusion equation. Such an equation is obtained (section 3.1) by adding appropriate dissipative and stochastic terms to the transport equation (30). The formal proof of covariance of the resulting stochastic Dirac equation will be given in section 3.2. Furthermore, we demonstrate that the stochastic Dirac equation preserves the norm of the state vector and we shall derive the corresponding covariant density-matrix equation (section 3.3). In order to have a unique, single-valued distribution

function of the stochastic process a certain integrability condition has to be satisfied. This condition will be derived and discussed in section 3.4. We construct in section 3.5 a stochastic Dirac equation which is covariant under the larger group of proper, orthochronous Lorentz transformations including spacetime translations. Finally, the localization properties of the stochastic process are discussed in section 3.6.

3.1. Equation of motion for the state vector

We start by rewriting equation (30) as

$$\mathrm{d}\psi = -\mathrm{i}V_\mu K(s)\psi \,\mathrm{d}s^\mu \tag{46}$$

where $\psi \equiv \psi(s, \boldsymbol{x})$, and $\mathrm{d}\psi \equiv \psi(s + \mathrm{d}s, \boldsymbol{x}) - \psi(s, \boldsymbol{x})$. Our aim is to turn equation (46) into a stochastic differential equation by adding a dissipative and a stochastic term to its right-hand side. In order to find these terms we demand that the resulting equation is Lorentz covariant and that it takes a form which is similar to the non-relativistic quantum-state-diffusion equation (2).

We are looking for an equation in the variables s^μ which label the hypersurfaces $\sigma(s)$. Therefore, we associate to each hypersurface $\sigma(s)$ a Lindblad operator $L(s)$ which acts on the corresponding state space $\mathcal{H}(s)$. Thus, for each Hilbert space in our Hilbert bundle we introduce a corresponding Lindblad operator $L(s)$, in the same way as we have introduced an operator $K(s)$, for example, for each Hilbert space in the preceding section.

Since each Hilbert space $\mathcal{H}(s)$ is equipped with a scalar product we can use this scalar product to define the adjoint $L^\dagger(s)$ of $L(s)$. Furthermore, we replace the expectation values occuring in the quantum-state-diffusion equation (2) by the expectation values

$$\langle L\rangle_s \equiv \langle\psi|L(s)|\psi\rangle_s \tag{47}$$

defined through the scalar product (15) in $\mathcal{H}(s)$. All that we have to do in order to end up with a covariant equation then is to construct a Lorentz invariant time parameter that replaces the time t of equation (2). This parameter is taken to be the Lorentz invariant scalar $a = \sqrt{s^2}$ whose differential is given by

$$\mathrm{d}a = n_\mu \mathrm{d}s^\mu. \tag{48}$$

This choice is motivated by the fact that a is a Lorentz invariant quantity formed by the 4-vector s^μ which reduces to the time coordinate x^0 for the equal time hypersurfaces given by $s = a(1, 0, 0, 0)$. With this choice the dissipative part of our stochastic Dirac equation will be proportional to $\mathrm{d}a$ and, in order to arrive at a closed equation of motion for the density matrix (see section 3.3), the stochastic part of that equation will be proportional to the differential $\mathrm{d}W(a)$ of a Lorentz invariant Wiener process. Thus, guided by the requirements of relativistic covariance we propose the following stochastic Dirac equation

$$\begin{aligned}\mathrm{d}\psi = {} & -\mathrm{i}V_\mu K(s)\psi \,\mathrm{d}s^\mu + n_\mu(\langle L^\dagger\rangle_s L(s) - \tfrac{1}{2}L^\dagger(s)L(s) - \tfrac{1}{2}\langle L^\dagger\rangle_s\langle L\rangle_s)\psi \,\mathrm{d}s^\mu \\ & + (L(s) - \langle L\rangle_s)\psi \,\mathrm{d}W(a)\end{aligned} \tag{49}$$

where $\mathrm{d}W(a)$ denotes the differential of a complex Wiener process with independent real and imaginary parts satisfying

$$\langle \mathrm{d}W(a)\rangle = 0 \qquad \langle \mathrm{d}W^*(a)\,\mathrm{d}W(a)\rangle = \mathrm{d}a \tag{50}$$

where the angular brackets denote the expectation value of the Wiener process. Equation (49) is our central result. It represents a relativistically covariant quantum-state-diffusion equation for a Dirac electron in an external electromagnetic field.

3.2. Proof of covariance

We shall give here the formal proof of the Lorentz covariance of the stochastic Dirac equation (49). To this end, consider two Lorentz frames A (coordinates x) and B (coordinates x') whose spacetime coordinates are connected by the Lorentz transformation (5). We then have the following transformation law

$$\psi'(s', \boldsymbol{x}') = S(\Lambda)\psi(s, \boldsymbol{x}(s', \boldsymbol{x}')) \equiv U(s', \Lambda)\psi(s, \boldsymbol{x}') \tag{51}$$

where $s'^{\mu} = \Lambda^{\mu}{}_{\nu}s^{\nu}$, and $S(\Lambda)$ denotes the usual spinor representation of the Lorentz group [11]. The function $\boldsymbol{x} = \boldsymbol{x}(s', \boldsymbol{x}')$ specifies the connection between the space coordinates $\boldsymbol{x}'$ and the space coordinates $\boldsymbol{x}$ of a point on that hypersurface which is given by s^{μ} in frame A and by s'^{μ} in frame B. This function is given by

$$x^i(s', \boldsymbol{x}') = (\Lambda^{-1})^i{}_0 x'^0(s', \boldsymbol{x}') + (\Lambda^{-1})^i{}_j x'^j. \tag{52}$$

The corresponding transformation law for the operator $L(s)$ takes the form

$$L'(s') = U(s', \Lambda)L(s)U^{\dagger}(s', \Lambda). \tag{53}$$

From the active point of view, the transformation (53) implies that the dissipative environment is transformed in the same way as the quantum object itself (the Dirac electron). To demonstrate that equation (49) is covariant under these transformation laws we first look at the transformation property of the scalar product (15). As is easily demonstrated we have

$$\langle\psi'|\psi'\rangle_{s'} = \langle\psi|\psi\rangle_s \tag{54}$$

which means that the operator $U(s', \Lambda)$ defined in (51) is unitary. In view of equation (53) we therefore obtain

$$\langle L'\rangle_{s'} = \langle L\rangle_s \qquad \langle L'^{\dagger}\rangle_{s'} = \langle L^{\dagger}\rangle_s. \tag{55}$$

Now, the differential of the wavefunction in frame B is

$$\mathrm{d}'\psi' = U(s', \Lambda)\,\mathrm{d}\psi + \frac{\partial U}{\partial s'^{\mu}}\psi\,\mathrm{d}s'^{\mu}. \tag{56}$$

Substituting $\mathrm{d}\psi$ by the right-hand side of equation (49) and using equations (51) and (53) we find

$$\begin{aligned}\mathrm{d}'\psi' = -\mathrm{i}\left\{U(s', \Lambda)V'_{\mu}K(s)U^{\dagger}(s', \Lambda) + \mathrm{i}\frac{\partial U}{\partial s'^{\mu}}U^{\dagger}\right\}\psi'\,\mathrm{d}s'^{\mu} \\ +(\langle L'^{\dagger}\rangle_{s'}L'(s') - \tfrac{1}{2}L'^{\dagger}(s')L'(s') - \tfrac{1}{2}\langle L'^{\dagger}\rangle_{s'}\langle L'\rangle_{s'})\psi'\,\mathrm{d}a' \\ +(L'(s') - \langle L'\rangle_{s'})\psi'\,\mathrm{d}W(a').\end{aligned}$$

Comparing this equation with (49) we immediately see that our stochastic Dirac equation is covariant if the following relation holds:

$$U(s', \Lambda)V'_{\mu}K(s)U^{\dagger}(s', \Lambda) + \mathrm{i}\frac{\partial U}{\partial s'^{\mu}}U^{\dagger} = V'_{\mu}K'(s') \tag{57}$$

where

$$K'(s') = \not{s}'(-\mathrm{i}\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}' + e\not{A}'(x'^0(s', \boldsymbol{x}'), \boldsymbol{x}') + m). \tag{58}$$

Equation (57) may be verified by an explicit calculation. Note however, that this equation expresses nothing but the covariance of our stochastic Dirac equation (49) for $L(s) \equiv 0$, i.e. the covariance of equation (30). The latter fact, in turn, is to be expected since (30) has been obtained from a Lorentz covariant equation, that is, from Dirac's equation (19).

Going through the above proof we see that covariance is, essentially, a consequence of the following facts.

(1) The dynamical variables s^μ form a 4-vector whose range is the interior of the forward light cone F_+ and is hence Lorentz invariant.

(2) The deterministic transport equation (30) is Lorentz covariant by construction.

(3) The transformation law for the Lindblad operators (53) together with the use of our Lorentz covariant scalar product (15) implies that the dissipative part of equation (49) transforms as a scalar.

(4) Since a is a Lorentz invariant quantity the Wiener differential $\mathrm{d}W(a)$ is a Lorentz invariant process.

3.3. Normalization and density-matrix equation

In order to find the change of the norm of the stochastic Dirac wavefunction we determine using the Ito calculus

$$\mathrm{d}\langle\psi|\psi\rangle_s = \langle \mathrm{d}\psi|\psi\rangle_s + \langle\psi|\mathrm{d}\psi\rangle_s + \langle \mathrm{d}\psi|\mathrm{d}\psi\rangle_s + \text{terms from derivative of scalar product.} \tag{59}$$

The terms that stem from the derivative of the scalar product are the last two terms on the right-hand side of equation (39) multiplied by $\mathrm{d}s^\mu$. Since we know already that these terms and the terms from the deterministic part of equation (49) cancel each other, we immediately get

$$\begin{aligned}\mathrm{d}\langle\psi|\psi\rangle_s &= \langle\psi|\langle L^\dagger\rangle_s L(s) + \langle L\rangle_s L^\dagger(s) - L^\dagger(s)L(s) - \langle L^\dagger\rangle_s\langle L\rangle_s|\psi\rangle_s\,\mathrm{d}a \\ &\quad +\langle\psi|(L^\dagger - \langle L^\dagger\rangle_s)(L - \langle L\rangle_s)|\psi\rangle_s\,\mathrm{d}a \\ &= 0.\end{aligned} \tag{60}$$

The covariant stochastic Dirac equation (49) yields a covariant equation of motion for the density matrix. The latter is defined in terms of the stochastic Dirac wavefunction by

$$\rho(s;\boldsymbol{x},\boldsymbol{y}) \equiv E\left\{\psi(s,\boldsymbol{x})\psi^\dagger(s,\boldsymbol{y})\gamma^0 \not{n}\frac{1}{n^0}\right\} \tag{61}$$

where E denotes the expectation value of the stochastic process. Defining the density matrix in this way we use the following definition for the trace of an operator $A(s)$,

$$\mathrm{tr}\,A(s) \equiv \int \mathrm{d}^3x\ \mathrm{tr}_{\mathrm{spin}}\,A(s;\boldsymbol{x},\boldsymbol{x}) \tag{62}$$

where $\mathrm{tr}_{\mathrm{spin}}$ denotes the trace over the spinor indices. On using this definition we obtain from (61)

$$\mathrm{tr}\,\rho(s) = E\left\{\int \mathrm{d}^3x\ \mathrm{tr}_{\mathrm{spin}}\,\psi(s,\boldsymbol{x})\psi^\dagger(s,\boldsymbol{x})\gamma^0\not{n}\frac{1}{n^0}\right\} = E\{\langle\psi|\psi\rangle_s\} = 1$$

and

$$\mathrm{tr}\{\rho(s)A(s)\} = E\{\langle A\rangle_s\}. \tag{63}$$

Moreover, definition (61) ensures that the density matrix $\rho(s)$ is Hermitian with respect to the scalar product in $\mathcal{H}(s)$.

On using equation (49) a straightforward calculation yields the following equations of motion for the density matrix,

$$\frac{\partial\rho}{\partial a} = -\mathrm{i}[H(s),\rho] + L(s)\rho L^\dagger(s) - \tfrac{1}{2}L^\dagger(s)L(s)\rho - \tfrac{1}{2}\rho L^\dagger(s)L(s) \tag{64}$$

$$(\delta^\nu_\mu - n_\mu n^\nu)\frac{\partial\rho}{\partial n^\nu} = -\mathrm{i}[W_\mu H(s),\rho]. \tag{65}$$

Equation (64) is a density matrix equation in Lindblad form. If we choose n as the unit vector in the direction of the time axis, $H(s)$ becomes equal to the Dirac Hamiltonian H_{D} and equation (64) takes on precisely the form of equation (1). The n-dependence of ρ is specified by equation (65).

3.4. Integrability condition of the process

We have seen in section 2.3 that the deterministic transport equation (30) is integrable. This is due to the fact that its solutions can be obtained by restricting the solutions of the Dirac equation (20) to the various hypersurfaces, as is expressed in equation (21). In other words, any solution $\psi(s, \boldsymbol{x})$ of the transport equation is really a function on the spacetime continuum. This is, in general, not true for the solutions of equation (49). Consider two different hypersurfaces $\sigma(s)$ and $\sigma(s')$ with some common point x, say. Then the values $\psi(s, \boldsymbol{x})$ and $\psi(s', \boldsymbol{x})$ of the wavefunctions on the different hypersurfaces at the common spacetime point do not, in general, coincide. Thus, the solutions of equation (49) are, in general, not wavefunctions on Minkowski space. However, what we do require is that the solution of (49) yields a probability distribution for the random wavefunctions which represents a function on the set Σ of flat, spacelike hypersurfaces. For this to be true a certain integrability condition has to be satisfied which will be derived in this section.

To begin with, we observe that equation (49) is not a stochastic differential equation for the wavefunction ψ in the usual sense. The reason is that equation (49) is an equation in four variables which form the 4-vector s^μ. To explain the meaning of equation (49) more precisely consider two points s_i and s_f within the forward light cone such that $a_f > a_i$ and a curve $s(\tau)$ that connects these points, i.e. with initial point s_i and endpoint s_f. We assume that $\mathrm{d}a > 0$ along $s(\tau)$. Without restriction we may then choose the parameter τ of the curve in such a way that $\mathrm{d}a = \mathrm{d}\tau$. Evaluated along such a curve equation (49) yields the following ordinary stochastic differential equation

$$\begin{aligned}\mathrm{d}_\tau\psi = -\mathrm{i}V_\mu K(s)\psi\frac{\mathrm{d}s^\mu}{\mathrm{d}\tau}\mathrm{d}\tau + \left(\langle L^\dagger\rangle_s L(s) - \frac{1}{2}L^\dagger(s)L(s) - \frac{1}{2}\langle L^\dagger\rangle_s\langle L\rangle_s\right)\psi\,\mathrm{d}\tau \\ + (L(s) - \langle L\rangle_s)\,\psi\,\mathrm{d}W(\tau)\end{aligned} \tag{66}$$

where $s \equiv s(\tau)$, $\psi \equiv \psi(\tau, \boldsymbol{x})$, and $\mathrm{d}_\tau\psi \equiv \psi(\tau + \mathrm{d}\tau, \boldsymbol{x}) - \psi(\tau, \boldsymbol{x})$. This is a stochastic differential equation for the process $\psi(\tau, \boldsymbol{x})$ in the usual sense and can be dealt with by the well-known Ito calculus. Thus, we see that equation (49) associates to each curve $s(\tau)$ a stochastic process $\psi(\tau, \boldsymbol{x})$ along this curve.

For example, if $\mathrm{d}s^\mu = n^\mu\mathrm{d}a$ is valid along the curve, that is, if $s(\tau)$ is a straight line with fixed direction n, the first term on the right-hand side in equation (66) becomes equal to $-\mathrm{i}H(s)\psi\,\mathrm{d}\tau$, where the Hermitian operator $H(s)$ is defined by equation (36). $H(s)$ reduces to the Dirac Hamiltonian if n is the unit vector in the direction of the time axis, $n = (1, 0, 0, 0)$. Thus, in this case the form of equation (66) is identical to that of equation (2).

It also follows from these considerations that in order for equation (49) to define a unique stochastic process $\psi(s, \boldsymbol{x})$ for all s, the following integrability condition must be satisfied. Suppose we have two different curves $s_1(\tau)$ and $s_2(\tau)$ of the above type that connect the given points s_i and s_f (see figure 2). Starting from the same initial conditions at the point s_i we then get, according to the above prescription, two random wavefunctions ψ_1 and ψ_2 at the common endpoint s_f; ψ_1 is obtained by following the stochastic evolution along $s_1(\tau)$ and ψ_2 by the stochastic evolution along $s_2(\tau)$. The integrability condition then states that the random wavefunctions ψ_1 and ψ_2 coincide in the sense that they have the

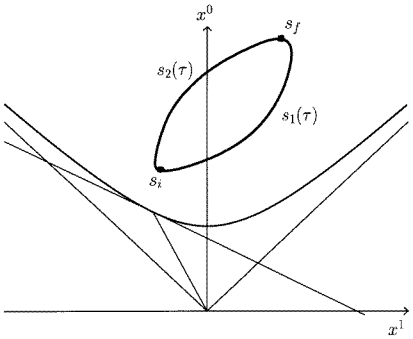

Figure 2. Graphical representation of the integrability condition for the stochastic process.

same probability distributions.

The integrability condition formulated above ensures that the probability distribution of $\psi(s, \boldsymbol{x})$ is a single-valued function of s, that is, a single-valued function on the whole interior of the forward light cone F_+. In particular, this condition implies that the density matrix equation is integrable. Given some initial condition on a specific hypersurface, this means that equations (64) and (65) have a unique solution $\rho(s)$ which is a function on the future light cone.

Mathematically, the integrability condition is expressed most conveniently by reformulating the stochastic Ito equation (49) by means of the corresponding Fokker–Planck equation for the probability density functional $P = P[s, \psi]$ which is given by

$$\frac{\partial}{\partial s^\mu} P[s, \psi] = \{\mathcal{L}_\mu(s) + n_\mu \mathcal{M}(s)\} P[s, \psi] \tag{67}$$

where we have introduced the first-order functional differential operator describing the deterministic drift

$$\mathcal{L}_\mu(s) = \mathrm{i} \int \mathrm{d}^3x \left(\frac{\delta}{\delta \psi_A(\boldsymbol{x})} [G_\mu \psi(\boldsymbol{x})]_A - \frac{\delta}{\delta \psi_A^*(\boldsymbol{x})} [G_\mu \psi(\boldsymbol{x})]_A^* \right) \tag{68}$$

($A = 1, \ldots, 4$ denotes the spinor index) and the second-order functional differential operator describing the diffusion process

$$\mathcal{M}(s) = \int \mathrm{d}^3x \int \mathrm{d}^3x' \, \frac{\delta^2}{\delta \psi_A(\boldsymbol{x}) \delta \psi_B^*(\boldsymbol{x}')} [(L - \langle L \rangle_s)\psi(\boldsymbol{x})]_A [(L - \langle L \rangle_s)\psi(\boldsymbol{x}')]_B^*. \tag{69}$$

In the above equations $\delta/\delta\psi_A(\boldsymbol{x})$ and $\delta/\delta\psi_A^*(\boldsymbol{x})$ denote functional Wirtinger derivatives [12, 13]. The differential equation described by $\mathcal{L}_\mu$ is given by

$$\begin{aligned} \frac{\partial}{\partial s^\mu} \psi(s, \boldsymbol{x}) &= -\mathrm{i} G_\mu \psi(s, \boldsymbol{x}) \\ &\equiv -\mathrm{i} V_\mu K(s) \psi + n_\mu (\langle L^\dagger \rangle_s L(s) - \tfrac{1}{2} L^\dagger(s) L(s) - \tfrac{1}{2} \langle L^\dagger \rangle_s \langle L \rangle_s) \psi. \end{aligned} \tag{70}$$

One can now express the integrability condition directly in terms of an equation for the probability distribution itself, namely by

$$\frac{\partial^2 P}{\partial s^\nu \partial s^\mu} = \frac{\partial^2 P}{\partial s^\mu \partial s^\nu}. \tag{71}$$

Inserting (67) into (71) we get

$$\left(\frac{\partial \mathcal{L}_\mu}{\partial s^\nu} - \frac{\partial \mathcal{L}_\nu}{\partial s^\mu} \right) P + (\mathcal{L}_\mu \mathcal{L}_\nu - \mathcal{L}_\nu \mathcal{L}_\mu) P$$

$$+n_\mu\left(\frac{\partial\mathcal{M}}{\partial s^\nu}-(\mathcal{L}_\nu\mathcal{M}-\mathcal{M}\mathcal{L}_\nu)\right)P-n_\nu\left(\frac{\partial\mathcal{M}}{\partial s^\mu}-(\mathcal{L}_\mu\mathcal{M}-\mathcal{M}\mathcal{L}_\mu)\right)P=0.$$

Setting equal to zero the first line in this equation we obtain a condition which is equivalent to the integrability condition for the deterministic drift (70). It is thus convenient to formulate the integrability condition as follows: (i) the deterministic drift of the process is integrable and (ii) the condition

$$n_\mu\left(\frac{\partial\mathcal{M}}{\partial s^\nu}-(\mathcal{L}_\nu\mathcal{M}-\mathcal{M}\mathcal{L}_\nu)\right)P-n_\nu\left(\frac{\partial\mathcal{M}}{\partial s^\mu}-(\mathcal{L}_\mu\mathcal{M}-\mathcal{M}\mathcal{L}_\mu)\right)P=0 \tag{72}$$

holds. By a lengthy but straightforward calculation and by using the integrability condition (45) of the transport equation one can show that the integrability condition for the deterministic drift is satisfied if the Lindblad operator $L(s)$ obeys the equation

$$\frac{\partial L(s)}{\partial s^\mu}=-\mathrm{i}[V_\mu K(s),L(s)]+n_\mu R(s) \tag{73}$$

where $R(s)$ can be any s-dependent operator. We now demonstrate that condition (73) also guarantees that (72) is satisfied. This shows then that (73) is a sufficient integrability condition.

Condition (72) means that for any functional $F=F[\psi]$ we must have

$$n_\mu E\left\{\frac{\partial\mathcal{M}^{\mathrm{ad}}}{\partial s^\nu}F+(\mathcal{L}_\nu^{\mathrm{ad}}\mathcal{M}^{\mathrm{ad}}-\mathcal{M}^{\mathrm{ad}}\mathcal{L}_\nu^{\mathrm{ad}})F\right\}$$
$$-n_\nu E\left\{\frac{\partial\mathcal{M}^{\mathrm{ad}}}{\partial s^\mu}F+(\mathcal{L}_\mu^{\mathrm{ad}}\mathcal{M}^{\mathrm{ad}}-\mathcal{M}^{\mathrm{ad}}\mathcal{L}_\mu^{\mathrm{ad}})F\right\}=0. \tag{74}$$

Note that the expectation value E of the process is defined explicitly in terms of the probability density functional as [12, 13]

$$E\{F\}\equiv\int\mathrm{D}\psi\mathrm{D}\psi^*P[s,\psi]F[\psi]. \tag{75}$$

The index 'ad' indicates the adjoint functional operator. Performing the functional differentiations we get after some algebra the following equation which is equivalent to equation (74),

$$n_\mu\int\mathrm{d}^3x\int\mathrm{d}^3x'\,E\left\{\frac{\delta^2F}{\delta\psi_A(\boldsymbol{x})\delta\psi_B^*(\boldsymbol{x}')}(Q_\nu\psi(\boldsymbol{x}))_A[(L-\langle L\rangle_s)\psi(\boldsymbol{x}')]_B^*\right\}$$
$$-n_\mu\int\mathrm{d}^3x\int\mathrm{d}^3x'\,E\left\{\frac{\delta^2F}{\delta\psi_A(\boldsymbol{x})\delta\psi_B^*(\boldsymbol{x}')}[(L-\langle L\rangle_s)\psi(\boldsymbol{x})]_A(Q_\nu\psi(\boldsymbol{x}'))_B^*\right\}$$
$$-(\mu\leftrightarrow\nu)=0 \tag{76}$$

where

$$Q_\nu\equiv\frac{\partial L(s)}{\partial s^\nu}+\mathrm{i}[V_\nu K(s),L(s)]-\left\langle\frac{\partial L(s)}{\partial s^\nu}+\mathrm{i}[V_\nu K(s),L(s)]\right\rangle_s. \tag{77}$$

On using equation (73) we observe that Q_ν is proportional to n_ν,

$$Q_\nu=n_\nu(R-\langle R\rangle_s). \tag{78}$$

Thus, we immediately see that (76) is fulfilled since the tensor $n_\mu n_\nu$ is symmetric.

We now demonstrate that our integrability condition (73) can always be fulfilled. To prove this we note first that by employing the (n,a)-parametrization of the hypersurfaces equation (73) can be decomposed into two separate equations,

$$(\delta_\mu^\nu-n_\mu n^\nu)\frac{\partial L(s)}{\partial n^\nu}=-\mathrm{i}[W_\mu H(s),L(s)] \tag{79}$$

and

$$\frac{\partial L(s)}{\partial a} = -\mathrm{i}[H(s), L(s)] + R(s). \tag{80}$$

Since the operator $R(s)$ is arbitrary we see from equation (80) that the a-dependence is not fixed by the integrability condition. This means that one can freely choose the a-dependence of $L(s) = L(n, a)$ corresponding to the physical situation under consideration (see the example below). To show that the integrability condition can always be fulfilled one therefore has to show that equation (79) is integrable and determines the n-dependence of $L(s)$. This can be done by noting that equation (73) is integrable for $R \equiv 0$. Employing equation (73) for $R \equiv 0$ we obtain

$$\frac{\partial^2 L}{\partial s^\nu \partial s^\mu} - \frac{\partial^2 L}{\partial s^\mu \partial s^\nu} = -\mathrm{i}\left[\frac{\partial (V_\mu K)}{\partial s^\nu} - \frac{\partial (V_\nu K)}{\partial s^\mu}, L\right] - [V_\mu K, [V_\nu K, L]] + [V_\nu K, [V_\mu K, L]]. \tag{81}$$

By means of the Jacobi identity this can be written as

$$\frac{\partial^2 L}{\partial s^\nu \partial s^\mu} - \frac{\partial^2 L}{\partial s^\mu \partial s^\nu} = -\mathrm{i}\left[\frac{\partial}{\partial s^\nu}(V_\mu K) - \frac{\partial}{\partial s^\mu}(V_\nu K) - \mathrm{i}[V_\mu K, V_\nu K], L\right] = 0 \tag{82}$$

where equation (45) has been used in the second equation. Thus, equation (73) is integrable for $R \equiv 0$. We can therefore solve this equation in the neighbourhood of each fixed a to get a solution of equation (79) for this a. Since the a-dependence is arbitrary as noted above, we then have completely fulfilled the integrability condition.

To give an example for the solution of equation (79) we consider a standard example from quantum optics, i.e. a two-level atom in the radiation field at zero temperature. For this case we have $L(s = a(1, 0, 0, 0)) = |\psi_g\rangle\langle\psi_e| = \sigma^-$, where $\psi_g(\boldsymbol{x})$ denotes the ground state and $\psi_e(\boldsymbol{x})$ the excited state. Denote by $\phi_{e,g}(x) = \exp(-\mathrm{i}\varepsilon_{e,g}x^0)\psi_{e,g}(\boldsymbol{x})$ the corresponding stationary solutions of the Dirac equation with time-independent Dirac Hamiltonian H_D. The operator $L(s)$ defined in the position representation by

$$L(s; \boldsymbol{x}, \boldsymbol{y}) = \mathrm{e}^{\mathrm{i}(\varepsilon_g - \varepsilon_e)a}\phi_g(x^0(s, \boldsymbol{x}), \boldsymbol{x})\phi_e^\dagger(x^0(s, \boldsymbol{y}), \boldsymbol{y})\gamma^0 \not{n}\frac{1}{n^0} \tag{83}$$

then fulfils the equation

$$\frac{\partial L(s)}{\partial s^\nu} = -\mathrm{i}[V_\nu K(s), L(s)] + \mathrm{i}(\varepsilon_g - \varepsilon_e)n_\nu L(s). \tag{84}$$

Multiplying this equation by the tensor $(\delta_\mu^\nu - n_\mu n^\nu)$ and summing over ν one gets equation (79). Thus, equation (83) represents the relativistic generalization of the negative frequency part σ^- of the atomic dipole operator which is widely used in quantum optical applications (see e.g. [14–16]).

The above example also indicates how to solve the integrability condition in the general case for a time-independent Dirac Hamiltonian: choose first a complete orthonormal set $\phi_\alpha(x) = \exp(-\mathrm{i}\varepsilon_\alpha x^0)\psi_\alpha(\boldsymbol{x})$ of stationary solutions of the Dirac equation corresponding to the energies ε_α. The solution of equation (79) which reduces for $s = (a, 0, 0, 0)$ to some given operator $L^{(0)}$ reads

$$L(s; \boldsymbol{x}, \boldsymbol{y}) = \sum_{\alpha,\beta} \mathrm{e}^{\mathrm{i}(\varepsilon_\alpha - \varepsilon_\beta)a}\phi_\alpha(x^0(s, \boldsymbol{x}), \boldsymbol{x})\langle\psi_\alpha|L^{(0)}|\psi_\beta\rangle\phi_\beta^\dagger(x^0(s, \boldsymbol{y}), \boldsymbol{y})\gamma^0 \not{n}\frac{1}{n^0}. \tag{85}$$

Since, as we have just seen, each term in the sum over α and β satisfies equation (79), equation (85) gives a solution of our integrability condition. On the equal-time hypersurfaces given by $s = (a, 0, 0, 0)$ we have $a = x^0$ and therefore

$$L((a,0,0,0);\boldsymbol{x},\boldsymbol{y}) = \sum_{\alpha,\beta} \psi_\alpha(\boldsymbol{x})\langle\psi_\alpha|L^{(0)}|\psi_\beta\rangle\psi_\beta^\dagger(\boldsymbol{y}) = L^{(0)}(\boldsymbol{x},\boldsymbol{y}) \tag{86}$$

as required.

3.5. Poincaré covariance

Up to now we have restricted ourselves to homogeneous Lorentz transformations which leave fixed the origin $x = 0$ of the coordinate system. The definition of the 4-vector s, of $a = \sqrt{s^2}$ and that of the corresponding hypersurface $\sigma(s)$ distinguishes a certain point, namely the origin of the coordinate system which serves as the base point of the future light cone F_+.

Without additional physical input the stochastic Dirac equation (49) is *not* covariant under spacetime translations due to this distinction of a specific point. However, a stochastic equation of motion which *is*, in fact, covariant under general inhomogeneous Lorentz transformations (Poincaré transformations)

$$x' = \Lambda x + y \tag{87}$$

can be obtained if one gives this distinguished point (which plays the role of the base point of the future light cone) an objective physical meaning as a certain spacetime event b and if one introduces the coordinates of b as parameters into the stochastic equation of motion as explained below. The same method is used in the representation theory of Poincaré-covariant quantum-dynamical semigroups for relativistic unstable particles [8].

Once such a point b has been fixed we may define $F_+(b)$ to be the future light cone based at this point b. The interior of $F_+(b)$ is defined by the conditions $(x-b)^2 > 0$ and $x^0 - b^0 > 0$. The former construction is recovered by setting $b = 0$. The flat, spacelike hypersurfaces $\sigma = \sigma(n, a, b)$ which cross $F_+(b)$ can then be parametrized uniquely by the equation

$$n^\mu(x_\mu - b_\mu) = a. \tag{88}$$

This implies that under Poincaré transformations (87) we have the following transformation laws

$$b' = \Lambda b + y \qquad n' = \Lambda n \qquad a' = a \qquad s' = \Lambda s. \tag{89}$$

Thus, according to their geometrical meaning n and s transform as true 4-vectors, a transforms as a scalar and b as a coordinate vector.

To define the spacetime point b we first remark that in order to describe a physical situation by means of a stochastic differential equation we need, of course, an initial condition which can be given by a pure or mixed state. In any case, according to the rules of quantum mechanics such an initial state requires an appropriate preparation measurement. The outcome of this measurement constitutes a classical event with certain spacetime coordinates. The classical event could be defined, for example, by the click in a particle detector, or by a certain meter position. We can then define b_μ as the coordinates of the classical event of the measurement which leads to the preparation of the initial state. The interior of the forward light cone $F_+(b)$ which defines the admissible hypersurfaces is then nothing but the absolute future of the classical measuring event b.

Starting now from the parametrization (88) and following the line of reasoning of sections 2 and 3.1 one is then immediately led again to the stochastic Dirac equation (49), where, however, the 4-vectors V_μ and W_μ have to be defined as

$$V_\mu \equiv \frac{1}{s^2}(2s_\mu - [x_\mu - b_\mu]) \qquad W_\mu \equiv s_\mu - [x_\mu - b_\mu]. \tag{90}$$

Note that W_μ is now a 4-vector operator which is tangent to the hypersurface $\sigma(n, a, b)$, that is, we have $n^\mu W_\mu = 0$. The stochastic Dirac equation (49) together with these definitions and the transformation laws (89) is obviously covariant under Poincaré transformations (87).

We emphasize that covariance under spacetime translations as formulated above is to be understood in a sense which differs from the usual one. Covariance in the usual sense means that one has a certain equation of motion which can be shown to be covariant *without reference to an initial condition*. In our formulation above, however, covariance under spacetime translations is guaranteed only if one introduces the initial measurement event b and the invariant future light cone originating from b into the stochastic equation of motion. Thus, a certain aspect of the initial condition, namely the coordinates of the measurement event (and not the initial state vector itself) enter the dynamical equation. Initial condition and stochastic time-evolution are therefore intimately connected in our theory and translational covariance holds if one transforms the whole equation of motion together with the invariant future light cone of the measurement event.

3.6. Relativistic localization

One of the most important features of quantum-state diffusion is the localization of dynamical variables induced by the dissipative and stochastic coupling to the environment [5, 6]. As an application of our relativistic formulation we shall derive in this section the relativistic equations governing the localization process.

For the dynamical observable we take the Linblad operator $L(s)$ itself and assume that $L(s)$ is self-adjoint and independent of a. This means that $L(s = (a, 0, 0, 0))$ is a time-independent Schrödinger operator defined on the equal-time hypersurfaces of a certain coordinate frame. We further assume that $L(s)$ commutes with the Hamiltonian $H(s)$ defined in equation (36),

$$[H(s), L(s)] = 0. \tag{91}$$

It follows then from (80) that the operator $R(s)$ is identically zero. Thus we have as a consequence of the integrability condition

$$\frac{\partial L(s)}{\partial s^\mu} = -\mathrm{i}[V_\mu K(s), L(s)]. \tag{92}$$

This equation represents the relativistic formulation of a quantum non-demolition condition. Such conditions are employed when discussing localization for so-called wide-open systems [6]. It is important to note that we cannot simply set $H_\mathrm{D} = 0$ as in the non-relativistic theory, for this would imply $K(s) = 0$ and the commutation relation (41) would be violated.

Using Ito calculus we obtain for any operator $A(s)$,

$$\begin{aligned}\mathrm{d}\langle A\rangle_s = \left\langle \frac{\partial A}{\partial s^\mu}\right\rangle_s \mathrm{d}s^\mu + \langle \mathrm{d}\psi|A|\psi\rangle_s + \langle\psi|A|\mathrm{d}\psi\rangle_s + \langle \mathrm{d}\psi|A|\mathrm{d}\psi\rangle_s \\ + \left(\int \frac{\mathrm{d}^3x}{s^0}\psi^\dagger\gamma^0\gamma_\mu A\psi - \frac{1}{s^0}\frac{\partial s^0}{\partial s^\mu}\langle A\rangle_s\right)\mathrm{d}s^\mu\end{aligned} \tag{93}$$

where the last two terms stem from the derivative of the scalar product (see section 2.3). By means of the stochastic Dirac equation (49) we find

$$\mathrm{d}\langle A\rangle_s = \left\langle \frac{\partial A}{\partial s^\mu} + \mathrm{i}[V_\mu K, A] \right\rangle_s \mathrm{d}s^\mu + \langle AT + T^\dagger A + M^\dagger AM\rangle_s \,\mathrm{d}a$$
$$+\langle AM\rangle_s \mathrm{d}W(a) + \langle M^\dagger A\rangle_s \,\mathrm{d}W^*(a) \tag{94}$$

where we have used the commutation relation (41) and introduced the abbreviations

$$\begin{aligned} T &\equiv \langle L^\dagger\rangle_s L(s) - \tfrac{1}{2}L^\dagger(s)L(s) - \tfrac{1}{2}\langle L^\dagger\rangle_s\langle L\rangle_s \\ M &\equiv L(s) - \langle L\rangle_s. \end{aligned} \tag{95}$$

Now, if the operator $A(s) = L(s)^k$ is some power of $L(s)$ we get on using equation (92) and (94)

$$\mathrm{d}\langle L^k\rangle_s = \langle L^k(L - \langle L\rangle_s)\rangle_s(\mathrm{d}W(a) + \mathrm{d}W^*(a)). \tag{96}$$

Taking the expectation value E of equation (96) for $k = 1$ and $k = 2$ one finds the following equations for the mean and dispersion of L,

$$\frac{\partial}{\partial s^\mu} E\{\langle L\rangle_s\} = 0 \qquad \frac{\partial}{\partial s^\mu}\,\mathrm{Var}(L) = 0 \tag{97}$$

where we have introduced the variance

$$\mathrm{Var}(L) \equiv E\{\langle L^2\rangle_s\} - (E\{\langle L\rangle_s\})^2 = \mathrm{tr}\{L^2\rho\} - (\mathrm{tr}\{L\rho\})^2. \tag{98}$$

Equation (97) states that the usual quantum expectation value and the variance of $L(s)$ are constant on all hypersurfaces.

Equation (96) also yields for $k = 1$

$$\mathrm{d}\langle L\rangle_s = \sigma^2(L)(\mathrm{d}W\,(a) + \mathrm{d}W^*\,(a)) \tag{99}$$

where we have introduced the dispersion of $L(s)$ in the state $\psi(s, \boldsymbol{x})$ by

$$\sigma^2(L) \equiv \langle L^2\rangle_s - \langle L\rangle_s^2. \tag{100}$$

Thus, as in the non-relativistic theory the self-diffusion of $L(s)$ is proportional to the dispersion of $L(s)$. Localization of the state vector dynamics can be described using the quantity

$$\mathrm{Var}_1(L) = E\{\sigma^2(L)\}. \tag{101}$$

This is the expectation value with respect to $P[s, \psi]$ of a 4th order moment of the wavefunction which cannot be expressed in terms of the density matrix. One finds on using equation (96)

$$\frac{\partial}{\partial a}\,\mathrm{Var}_1(L) = -2E\{[\sigma^2(L)]^2\} \tag{102}$$

and, equivalently,

$$\frac{\partial}{\partial s^\mu}\,\mathrm{Var}_1(L) = -2n_\mu E\{[\sigma^2(L)]^2\}. \tag{103}$$

Equations (102) and (103) represent the covariant equations governing the dynamical localization in our relativistic theory. Since it follows from equation (102) that

$$\frac{\partial}{\partial a}\,\mathrm{Var}_1(L) \leqslant -2[\mathrm{Var}_1(L)]^2 \tag{104}$$

we can conclude that $\mathrm{Var}_1(L)$, starting from some (positive) value at an initial value of a, decreases to zero for increasing a. Note that (101) is the expectation of the dispersion of

$L(s)$ in the state $\psi(s, \boldsymbol{x})$. Therefore, if the value $\mathrm{Var}_1(L) = 0$ is finally reached, the state vector ends up with probability 1 in an eigenspace of $L(s)$. This is just the state-vector localization to the eigenspaces of the dynamical observable $L(s)$.

We finally remark that the quantum variance $\mathrm{Var}(L)$ can be written as a sum of two variances [17]

$$\mathrm{Var}(L) = \mathrm{Var}_1(L) + \mathrm{Var}_2(L) \tag{105}$$

where $\mathrm{Var}_1(L)$ is defined in (101) and $\mathrm{Var}_2(L)$ is given by

$$\mathrm{Var}_2(L) \equiv E\{\langle L\rangle_s^2\} - (E\{\langle L\rangle_s\})^2. \tag{106}$$

Since $\psi(s, \boldsymbol{x})$ is a random variable, $\langle L\rangle_s$ is a real random number. $\mathrm{Var}_2(L)$ is therefore the statistical variance of the random number $\langle L\rangle_s$. Since $\mathrm{Var}_1(L)$ decreases monotonously to zero for increasing a, and since $\mathrm{Var}(L)$ stays constant, it follows that $\mathrm{Var}_2(L)$ increases monotonously until it reaches the value $\mathrm{Var}_2(L) = \mathrm{Var}(L)$. This shows that in the final state of the process the quantum statistical variance $\mathrm{Var}(L)$ is equal to the statistical fluctuations of the stochastic variable $\langle L\rangle_s$ measured by $\mathrm{Var}_2(L)$.

4. Summary and conclusions

In this paper we have generalized the quantum-state-diffusion model to relativistic quantum mechanics. This has been achieved by introducing the set Σ of flat, spacelike hypersurfaces which cross the future light cone F_+ and associating with each hypersurface $\sigma \in \Sigma$ a scalar product and a Hilbert space of Dirac wavefunctions. On the basis of these constructions a stochastic Dirac equation for an electron in a dissipative environment has been formulated. This equation takes the form of a direct generalization of the equation of motion of the quantum-state-diffusion model.

We have discussed several properties of our stochastic Dirac equation. First, we have derived the corresponding density matrix equation and the normalization of the state vector. Further, the integrability condition has been derived which guarantees that the stochastic process has a unique probability density functional which is single-valued on the whole interior of the future light cone. It has been demonstrated that the integrability condition can always be satisfied and some examples of its solution have been discussed. The extension of the formalism to include Poincaré covariance has been given. Finally, some aspects of the localization properties of the relativistic theory have been discussed.

Concluding, let us emphasize the main advantages of our relativistic fomulation. First, by formulating the stochastic process as a process for wavefunctions defined on the hypersurfaces, that is, as a process in the Hilbert bundle, our stochastic Dirac equation is of great generality. Since, as we have demonstrated, the integrability condition for the process can always be solved, we do not have to impose any restriction on the Lindblad operator. Thus, the Lindblad operator can be any local or non-local operator and the generality of the non-relativistic theory is maintained in our formulation. This point is essentially what distinguishes our approach from that of Ghirardi *et al* [18]. These authors have developed a stochastic state-vector theory for relativistic quantum fields. Their approach is more general in the respect that they treat relativistic many-particle systems (although, as it seems, in their theory difficulties with infinities are more serious than those encountered in usual quantum-field theory and it is not clear whether their approach yields a renormalizable theory). However, the approach of [18] is more special than our formulation in the respect that they treat only interactions involving local fields and use fluctuating terms represented by a local white noise process in spacetime.

The close formal analogy of our stochastic Dirac equation to the equation of quantum-state diffusion further allows the direct translation of many results of the non-relativistic theory to the relativistic one. As an example for this fact we have discussed the localization properties of the stochastic process.

Finally, the formal setting developed in this paper also enables us to formulate relativistically covariant piecewise deterministic jump processes. This is an important point since piecewise deterministic processes have been used to describe continuous measurements in quantum optics [19, 20]. The formalism proposed here therefore allows the development of a relativistically covariant stochastic theory of continuous measurements of open quantum systems.

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